



5.4 Orthogonality

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Introduction

- In this topic, we will
 - Define orthogonality
 - Describe its properties





Orthogonality

- Previously, we claimed that for non-zero vectors $\mathbf{u}$ and $\mathbf{v}$,

$$\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|_2 \|\mathbf{v}\|_2} = \cos(\theta)$$

- Note that if the inner product is zero,

$$\text{then } \cos(\theta) = 0, \text{ so } \theta = \pm \frac{\pi}{2}$$

– That is, it would appear that $\mathbf{u}$ and $\mathbf{v}$ are at right angles

- Thus, we will say that if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$,
the vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal*





Orthogonality

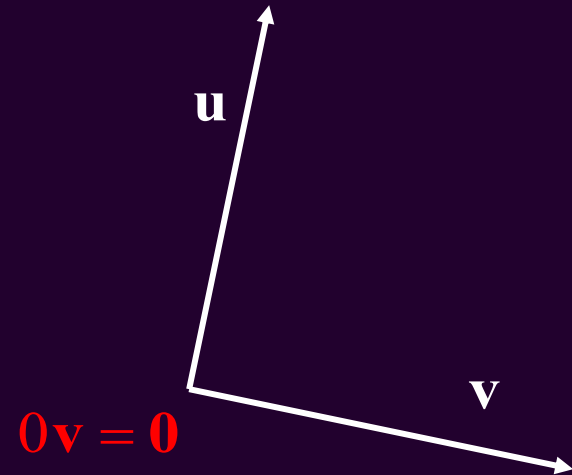
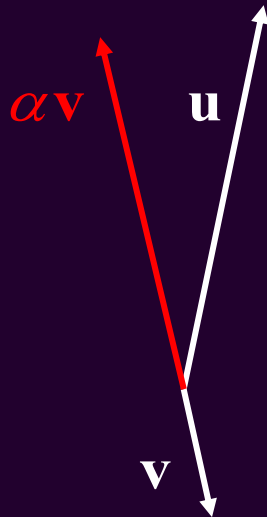
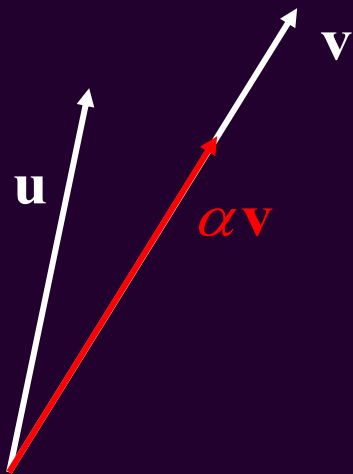
- The word *orthogonal* comes from two Greek words:
ortho for “right”
gon for “angle”
- Thus, to be orthogonal is
to have the property of being at right angles
- If two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, we may just write $\mathbf{u} \perp \mathbf{v}$





Orthogonality

- What does it mean for two vectors to be orthogonal?



- Two vectors are orthogonal if neither vector contains any information about the other





Examples

- Consider the following: $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

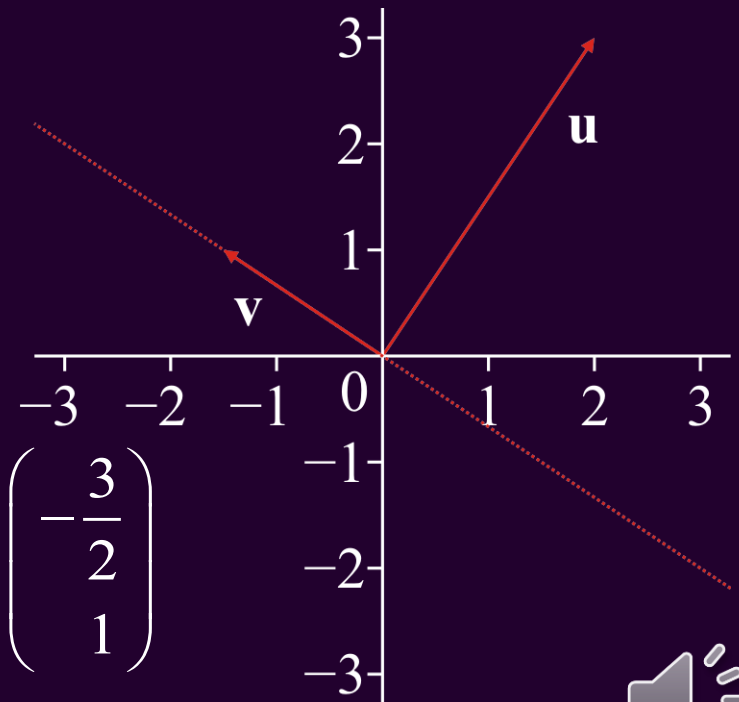
– Thus, for these to be orthogonal, $\langle \mathbf{u}, \mathbf{v} \rangle = 2v_1 + 3v_2 = 0$

$$2v_1 = -3v_2$$

$$v_1 = -\frac{3}{2}v_2$$

– Thus, $\mathbf{v} = \begin{pmatrix} -\frac{3}{2}v_2 \\ v_2 \end{pmatrix}$

– But this is all scalar multiples $\mathbf{v} = v_2 \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix}$





Examples

- For two dimensional vectors,
it's really easy to find an orthogonal vector

– Given $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$,

all scalar multiples of $\begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$ are orthogonal to $\mathbf{u}$





The zero vector

- Regardless of which vector space $\mathcal{V}$ we are in,
if $\mathbf{u} \in \mathcal{V}$, then $\langle \mathbf{u}, \mathbf{0} \rangle = 0$
 - That is, the zero vector is orthogonal
to all vectors in that vector space

Theorem

The only vector orthogonal to all vectors is the zero vector.

Proof:

Suppose $\mathbf{u} \in \mathcal{V}$ has the property that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in \mathcal{V}$.

Therefore, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$.

But $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$. ■





Summary

- Following this topic, you now
 - Understand the term orthogonal
 - Have an idea as to how we will use orthogonality





References

- [1] <https://en.wikipedia.org/wiki/Orthogonality>





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

<https://www.rbg.ca/>

for more information.





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